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Journal of Sound and Vibration 265 (2003) 1111–1115

JOURNAL OF SOUND AND VIBRATION

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Letter to the Editor

On a determinant formula used for the derivation of frequency equations of combined systems

M. Gürgöze*

Faculty of Mechanical Engineering, Technical University of Istanbul, 80191 Gümüşsuyu, Istanbul, Turkey

Received 22 October 2002; accepted 4 December 2002

1. Introduction

In Ref. [1], Cha and Wong developed a new approach for the derivation of the characteristic equation of a system consisting of a continuous structure to which several spring–mass systems (say s) are attached. After discretizing the continuous structure (say to n d.o.f) the corresponding generalized eigenvalue problem of order $(n \times n)$ is formulated whose stiffness and mass matrices consist of diagonal matrices modified by a total of s rank-one matrices. The generalized eigenvalue problem is manipulated such that the eigenvalues can be obtained by solving a much smaller characteristic determinant of order $(s \times s)$, each element of which involves a sum of n terms, instead of finding the roots of a much larger determinant of order $(n \times n)$.

Cha and Pierre applied the above approach successfully to various vibrational systems in Refs [2–5], whereas the present author made use of it for obtaining the characteristic equation of proportionally damped systems subjected to damping modifications [6].

From the mathematical point of view, what is done in Ref. [1] is essentially the derivation of a formula for the determinant of a matrix which is the sum of a diagonal matrix and several dyadic products. After noting that the proof of the new form of the characteristic equation is rather lengthy, its proof is given in the appendix for $s = 1$, i.e., for the special case of only one spring–mass system. Then it is stated that the general case of arbitrary s is merely an extension of the given derivation. The present author has made in Ref. [7] a comment on the formula obtained for $s = 1$. The basis of the derivation for $s = 1$ and then for various s values are elementary determinant equations like, multiplication of various columns by appropriate factors and then summation or subtraction from each other.

In the next section, the proof of a more general formula will be given which enables one to establish a simple formula for the determinant of the sum of a regular square matrix (not necessarily diagonal) and several dyadic products.

*Corresponding author.
E-mail address: gurgozem@itu.edu.tr (M. Gürgöze).

0022-460X/03/$ - see front matter
doi:10.1016/S0022-460X(02)01624-3

2. Theory

Start with the well-known determinant formula [8,9]

$$\det\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} = \det\mathbf{A}\det(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}), \tag{1}$$

where it is assumed that $\mathbf{A}$ is regular, i.e., $\det\mathbf{A}$ not equal to zero. The matrix $(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})$ is referred to as the Schur complement of $\mathbf{A}$ in the block matrix at the left side of Eq. (1), [9]. Here, the dimensions of the submatrices are as follows: $\mathbf{A}$ $(n\times n)$, $\mathbf{B}$ $(n\times m)$, $\mathbf{C}$ $(m\times n)$ and $\mathbf{D}$ $(m\times m)$.

In case of regular $\mathbf{D}$, the determinant of the block matrix can also be expressed as

$$\det\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} = \det\mathbf{D}\det(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}). \tag{2}$$

Now substitute

$$\mathbf{A}=\mathbf{A},\quad \mathbf{B}=-\mathbf{X},\quad \mathbf{C}=\mathbf{Y}^{\mathrm{T}},\quad \mathbf{D}=\mathbf{I}_m, \tag{3}$$

where $\mathbf{I}_m$ denotes the m-dimensional unit matrix and $\mathbf{X}$ and $\mathbf{Y}$ are new $n\times m$ matrices. Then, making use of Eqs (1) and (2)

$$\det\begin{bmatrix}\mathbf{A} & -\mathbf{X}\\ \mathbf{Y}^{\mathrm{T}} & \mathbf{I}_m\end{bmatrix} = \det\mathbf{A}\det(\mathbf{I}_m+\mathbf{Y}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{X}) = \det\mathbf{I}_m\det(\mathbf{A}+\mathbf{X}\mathbf{Y}^{\mathrm{T}}) \tag{4}$$

can be written. Hence, the formula

$$\det(\mathbf{A}+\mathbf{X}\mathbf{Y}^{\mathrm{T}}) = \det\mathbf{A}\det(\mathbf{I}_m+\mathbf{Y}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{X}) \tag{5}$$

is obtained. The special case $\mathbf{X}=\mathbf{x}$, $\mathbf{Y}^{\mathrm{T}}=\mathbf{y}^{\mathrm{T}}$ where $\mathbf{x}$ and $\mathbf{y}$ denote $(n\times 1)$ matrices, i.e., column vectors, yields

$$\det(\mathbf{A}+\mathbf{x}\mathbf{y}^{\mathrm{T}}) = \det\mathbf{A}\det(1+\mathbf{y}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{x}), \tag{6}$$

which is a well-known formula for the determinant of a regular square matrix modified by a rank-one matrix. This formula is often used in control theory in the context of multivariable feedback and pole location [10].

Let

$$\mathbf{X}=[\mathbf{x}_1,\ldots,\mathbf{x}_m],\quad \mathbf{Y}=[\mathbf{y}_1,\ldots,\mathbf{y}_m]. \tag{7}$$

Then it can be shown that

$$\sum_{i=1}^{m}\mathbf{x}_i\mathbf{y}_i^{\mathrm{T}} = \mathbf{X}\mathbf{Y}^{\mathrm{T}}. \tag{8}$$

On the other hand, via the definitions in Eq. (7), the triple matrix product on the right side of formula (5) can be written as

$$\bar{\mathbf{G}} = \mathbf{Y}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{X} = \begin{bmatrix} \mathbf{y}_1^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{x}_1 & \cdots & \mathbf{y}_1^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{x}_j & \cdots & \mathbf{y}_1^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{x}_m \\ \vdots & & \vdots & & \vdots \\ \mathbf{y}_i^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{x}_1 & \cdots & \mathbf{y}_i^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{x}_j & \cdots & \mathbf{y}_i^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{x}_m \\ \vdots & & \vdots & & \vdots \\ \mathbf{y}_m^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{x}_1 & \cdots & \mathbf{y}_m^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{x}_j & \cdots & \mathbf{y}_m^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{x}_m \end{bmatrix}. \tag{9}$$

Finally, the following formula is obtained:

$$\det\left(\mathbf{A} + \sum_{i=1}^{m} \mathbf{x}_i \mathbf{y}_i^{\mathrm{T}}\right) = \det \mathbf{A} \det(\mathbf{I}_m + \bar{\mathbf{G}}), \tag{10}$$

where the matrix $\bar{\mathbf{G}}$ is defined in Eq. (9) in terms of the column vectors $\mathbf{x}_i$, $\mathbf{y}_i$ and the inverse of the matrix $\mathbf{A}$.

The above formula enables one to express the determinant of a regular square matrix modified by a total of m rank-one matrices in terms of the sum of a special $(m \times m)$ matrix $\bar{\mathbf{G}}$ and the m-dimensional unit matrix.

In order to compare the formula developed in this study with Eq. (9) in Ref. [1], let it be assumed

$$\begin{aligned} &\mathbf{A} = (\mathbf{K}^d - \omega^2\mathbf{M}^d), \quad \mathbf{x}_i = \sigma_i \mathbf{\Phi}_i, \quad \mathbf{y}_i = \mathbf{\Phi}_i, \\ &\mathbf{\Phi}_i = \left[\phi_1(x), \ldots, \phi_n(x)\right]^{\mathrm{T}}\Big|_{x=x_i}, \quad s = m, \end{aligned} \tag{11}$$

where $\mathbf{K}^d$ and $\mathbf{M}^d$ denote the diagonal stiffness, and mass matrices and ω is the eigenfrequency of the combined system resulting from the discretized continuous structure to which m spring–mass systems are attached. $\mathbf{\Phi}(x)$ represents the $(n \times 1)$ vector whose elements $\phi_j(x)$ $(j = 1,2,\ldots,n)$ are the corresponding eigenfunctions of the unconstrained structure, used for the discretization procedure.

With the definitions in Eq. (11), Eq. (10) gives

$$\det\left(\mathbf{K}^d - \omega^2\mathbf{M}^d + \sum_{i=1}^{m} \sigma_i \mathbf{\Phi}_i \mathbf{\Phi}_i^{\mathrm{T}}\right) = \det(\mathbf{K}^d - \omega^2\mathbf{M}^d)\det \mathbf{G}, \tag{12}$$

where the matrix $\mathbf{G} = \mathbf{I}_m + \bar{\mathbf{G}}$ is defined using Eqs. (9) and (11) as

$$\mathbf{G} = [g_{ij}] = [\delta_i^j + \mathbf{\Phi}_i^{\mathrm{T}}(\mathbf{K}^d - \omega^2\mathbf{M}^d)^{-1}\sigma_j\mathbf{\Phi}_j] \tag{13}$$

δ_i^j being the the Kronecker delta.

The (i,j)th element of $\mathbf{G}$, i.e., g_{ij} can be written as

$$g_{ij} = \delta_i^j + \sigma_j \sum_{r=1}^{n} \frac{\phi_r(x_i)\phi_r(x_j)}{(K_r - \omega^2 M_r)} \quad (i,j = 1, \ldots, m), \tag{14}$$

noting that the (r,r)th elements of the diagonal matrices $\mathbf{K}^d$ and $\mathbf{M}^d$ are denoted in Ref. [1] as K_r and M_r, respectively.

The matrix element g_{ij} above differs from b_{ij} in Ref. [1] by the factor σ_j .The reason is explained below.

In order to obtain the frequency equation of the combined structure made up of the continuous structure and the m spring–mass systems attached to it, one has to equate the right side of Eq. (12) to zero which yields

$$\det[g_{ij}] = 0. \tag{15}$$

The validity of this equation holds further if the first column of the determinant is divided by σ_1, the second by σ_2 and finally, the last column by σ_m, which leads to

$$\det[g'_{ij}] = 0, \tag{16}$$

where

$$g'_{ij} = \frac{1}{\sigma_j}\delta_i^j + \sum_{r=1}^{n} \frac{\phi_r(x_i)\phi_r(x_j)}{(K_r - \omega^2 M_r)}, \quad (i,j = 1, \ldots, m). \tag{17}$$

Comparison of expression (17) with expression (9) in Ref. [1] reveals that, actually

$$g'_{ij} = b_{ij}. \tag{18}$$

3. Conclusions

This study is concerned with the derivation of a formula which enables one to obtain the determinant of the sum of a regular square matrix and several dyadic products. The present formula is much easier to prove although it is more general than the recently developed one which can be employed only for a diagonal matrix $\mathbf{A}$, rather than a general square and regular matrix. Both forms of the formula can be used successfully for obtaining the characteristic equations of continuous structures to which several spring–mass systems are attached.

Acknowledgements

The author expresses his gratitude to his friend Professor P.C. Müller from University of Wuppertal (Germany) for his help in obtaining Eq. (10)

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